Some remarks on the question of charge densities in stationary–current–carrying conductors

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Abstract. Recently, some discussions arose as to the definition of charge and the value of the density of charge in stationary–current–carrying conductors. We stress that the problem of charge definition comes from a misunderstanding of the usual definition. We provide some theoretical elements which suggest that positive and negative charge densities are equal in the frame of the positive ions.

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1. Introduction

Recently Ivezić [1–4] has questioned the correctness of the usual charge definition and raised the problem of the charge density inside a current–carrying conductor. He gives a new definition of charge and suggests that the ion charge density, as measured in the ions reference frame, could be equal to the electron charge density as measured in the electrons reference frame. This is contrary to the usual view that these two charge densities are equal in the ions reference frame. A consequence of Ivezić's assumption is that outside a conducting wire there should be a static electric field when a stationary current is flowing in the conductor. This author also claims that in the seventies some experiments were carried out [5–7] (actually also recently in [8,9]) which are in agreement with this new definition of charge. As a consequence of Ivezić's papers, many authors [10–13] have taken up a definite position against his ideas which provoked a discussion which, in our opinion, has not yet settled down the question. The paper is organized as follows. In sect. 2 we show that the usual charge definition is the correct one; in sect. 3 we show how to deal correctly with invariant integrals, while in sect. 4 we disprove Ivezić's hypothesis of the non–zero electric field outside current–carrying conductors.

2. Independence of the charge on the velocity

Purcell [14] had pointed out that the independence of the charge from the velocity means that the integral

$$\frac{1}{4\pi}\int_{A(t)} \boldsymbol{E}\, d\boldsymbol{a} = Q \tag{2.1}$$

does not depend on the motion of the particles inside the closed surface $A(t)$; moreover, if we choose another closed surface which contains the same number of particles, then the value of the flux of the electric field from this surface is the same. If this happens for an inertial reference frame then, because of the principle of relativity, it must be true for any other inertial frame. Therefore, if we consider in a frame O a system of charges at the time t in a volume surrounded by the closed surface $A(t)$ and in a frame O' in motion with constant velocity with respect to O any volume which contains at the time t' the same particles surrounded by the closed surface $A'(t')$, then

$$\int_{A(t)} \boldsymbol{E}\ d\boldsymbol{a} = \int_{A'(t')} \boldsymbol{E}'\ d\boldsymbol{a}' \tag{2.2}$$

where the two integrals are evaluated at the times t and t' respectively. We want to point out that eq. (2.2) does not give a recipe to obtain the closed surface $A'(t')$ nor the time at which to perform the integration once $A(t)$ has been given. Only for a closed system inside a surface A (here with closed system we mean that no charges cross its boundaries) it is correct to take the surface, for instance, as simultaneously seen in O' and perform the integration in any instant of time t'. In this case in fact the problem is independent of time. On the contrary, if we consider a non–closed system, like a piece of wire with a steady current, then it is not trivial to choose a correct surface in O' (a possible wayout of this difficulty has been proposed in [10]).

Purcell in his definition of invariance of charge does only state that equation (2.2) must hold if, and only if, the boundaries in the two reference frames contain the same particles. And this is equivalent to admit the postulate that the charge of a particle does not depend on its motion. These specifications are to show the right way in which eq. (2.2) should be interpreted. In fact we think that in [1, 4] there has been a misunderstanding about the meaning and validity of equation (2.2).

We want to stress that the invariance of the charge as defined in (2.2), *i.e.* its independence from motion, is a consequence of the fact that the equation of charge conservation is a continuity equation

$$\frac{\partial \rho}{\partial t} + div\boldsymbol{j} = 0 \ , \tag{2.3}$$

fulfilled in every inertial frame.

To show this let us consider a charge at rest in the origin of an inertial frame O. At the time $t = 0$ we turn on a suitable field of force which causes an acceleration along the x direction for a time τ. Therefore, at the time $t = \tau$ the charge is moving with a uniform velocity V in the x direction. Now let us consider a closed spherical surface with centre at the origin and radius r_0 large enough not to be crossed by the charge for an interval of time $I = [\tau, T]$ (where $T > \tau$). If we integrate eq. (2.3) in this reference frame on the volume enclosed by that surface for every $t < T$ we obtain

$$\frac{dQ}{dt} = \frac{d}{dt}\int \rho dV = -\int \boldsymbol{j} \ d\boldsymbol{a} = 0 \tag{2.4}$$

This means that the total charge inside a close surface is not affected by the state of motion but can vary if and only if some charged particles cross the boundary.

This reasoning can be done for any number of charges and any initial configuration. Therefore (2.2) (in the sense and with the specification we have made) derives from the assumption of equation (2.3) for the conservation of charge.

All these considerations show that a correct way to think about charge conservation is the continuity equation. According to this equation, it is not important that the charge in a certain volume does not change, but what matters is the fact that the change should be only due to the charges that cross the closed surface which is the boundary of that volume.

3. The invariant integral of charge

If we want to consider a relativistic scalar quantity, that is something which is invariant under Lorentz transformation, eq. (2.2) becomes useless first because the integrand is not written as a relativistic scalar and second because there is no a priori relation between $A(t)$ and $A'(t')$. One thing is to say that equation (2.2) holds and another one is to construct an invariant quantity. The first statement means that the charge does not depend on the velocity. The second means that if we want to find, in a relativistic invariant way, the same amount of charges in two different reference frames we must also take into account the fact that the difference of simultaneity gives rise to differences as to the charges contained contemporaneously in a volume [15]. To obtain such an invariant quantity we must consider as integrand a relativistic scalar and a hypersurface as a domain. In this way if one changes the inertial frame, the value of the integral remains the same, but its meaning in the two frames is in general different because of the difference of simultaneity. As pointed out by [11] the correct invariant quantity is that given in [16], that is

$$Q = \frac{1}{c} \int_H j^\mu dS_\mu \tag{3.1}$$

where H is a hypersurface. In this way Q turns out to be the sum of those charges the world lines of which cross this hypersurface. When one has to do with integrals over hypersurfaces, the way to handle them is to parametrize the domain. In the four–dimensional space–time of special relativity this means that we must consider the coordinates as functions of three real parameters, that is to say

$$\begin{aligned} &x^\mu = \phi^\mu(u_i) \qquad\qquad u_i \in U_i \subset \Re \\ &i \in 1, 2, 3 \end{aligned} \tag{3.2}$$

Then by definition

$$\int_H j^\mu(x^\nu) dS_\mu = \int_{U_1} du_1 \int_{U_2} du_2 \int_{U_3} du_3 \; j^\mu(u_i) \; n_\mu \tag{3.3}$$

where (putting $\phi = (\phi^0, \phi^1, \phi^2, \phi^3,)$ and $\phi_{u_i} = \partial\phi/\partial u_i$),

$$n^\mu = (\phi_{u_1} \wedge \phi_{u_2} \wedge \phi_{u_3})^\mu = -\frac{1}{6} e^{\mu\alpha\beta\gamma} D_{\alpha\beta\gamma} \tag{3.4}$$

and

$$D^{\alpha\beta\gamma} = det \begin{pmatrix} \phi^\alpha_{u_1} & \phi^\beta_{u_1} & \phi^\gamma_{u_1} \\ \phi^\alpha_{u_2} & \phi^\beta_{u_2} & \phi^\gamma_{u_2} \\ \phi^\alpha_{u_3} & \phi^\beta_{u_3} & \phi^\gamma_{u_3} \end{pmatrix} \tag{3.5}$$

If we perform a Lorentz transformation, then one has

$$\begin{gathered} x'^\mu = l^\mu_\nu x^\nu = l^\mu_\nu \phi^\nu(u_i) = \psi^\mu(u_i) \\ j^\mu[\phi^\nu(u_i)] n_\mu(u_i) = j'^\mu[\psi^\nu(u_i)] n'_\mu(u_i) \end{gathered} \tag{3.6}$$

In this way we have that (putting $x' = (x'^\mu) = (l^\mu_\nu x^\nu) = \boldsymbol{l}x$)

$$\begin{aligned} \int_H j^\mu(x^\nu)\ dS_\mu = \int_{U_1} du_1 \int_{U_2} du_2 \int_{U_3} du_3\ j^\mu[\phi(u_i)]\ n_\mu = \\ = \int_{U_1} du_1 \int_{U_2} du_2 \int_{U_3} du_3\ j'^\mu[\psi(u_i)]\ n'_\mu = \int_{\boldsymbol{l}(H)} j'^\mu(x'^\nu)\ dS'_\mu \end{aligned} \tag{3.7}$$

so that integral (3.3) is invariant under Lorentz transformation.

Another way to consider integral (3.1) is by means of differential forms. In fact let us consider j_μ as the components of the 1–form

$$\boldsymbol{J} = j_\mu\ dx^\mu \tag{3.8}$$

From (3.8) we can define the dual 3–form ${}^*\boldsymbol{J}$:

$$\begin{gathered} {}^*\boldsymbol{J} = \frac{1}{6}\ j_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \\ j_{\alpha\beta\gamma} = e_{\alpha\beta\gamma\mu} j^\mu \end{gathered} \tag{3.9}$$

where $e_{\alpha\beta\gamma\mu}$ is the totally antisymmetric tensor as defined, for instance, in paragraph 6 of ref. [16]. With these definitions integral (3.1) can be written as (see, for instance, Box 4.1 D in ref. [17])

$$Q = \int_H {}^*\boldsymbol{J} \tag{3.10}$$

If we consider the exterior derivative of ${}^*\boldsymbol{J}$ we see that it is zero because of the continuity equation (2.3); that is to say ${}^*\boldsymbol{J}$ is the exterior derivative of a 2–form. This 2–form is the dual of the tensor of the electromagnetic field $\boldsymbol{F} = F_{\mu\nu} dx^\mu \wedge dx^\nu$:

$$\boldsymbol{d}{}^*\boldsymbol{F} = \frac{4\pi}{c}{}^*\boldsymbol{J} \tag{3.11}$$

This is very important in the calculation of (3.10). In fact this implies (as in the one–dimensional case when one deals with an exact 1–form) that the value of the integral is left unchanged when, in order to simplify the calculatins, we deform the hypersurface of the domain leaving the boundaries unaltered.

This way to perform calculation is equivalent to the one used in [11]. The only difference is that Bilić in [11] considers a two dimensional space–time. In this particular situation the integrand is an exact differential form (because of the continuity equation) and so the value of the integral is the same for any path of integration between the points A and B which must be the same contrary to what has been said in [4]. In other words, this means that in [4] Ivezić does not interpret correctly eq (3.1).

The way in which we have considered integral (3.1) implies automatically that we are dealing with the same total carge as well with the same charged particles in the two reference frames. The root of this lies in the continuity equation. In fact, on the one hand, one has to remember (as pointed out by Ivezić in [4]) that j^μ is a four–vector because of the postulate of relativity that imposes the covariance of continuity equation. In this way $j^\mu\ dS_\mu$ is a scalar (and therefore we are dealing with the same charged particles). On the other hand, if we have the same particles, the continuity equation implies that we have also the same total charge, independent on their state of motion.

In this way we have stressed once more that, as pointed out by paragraph 29 of ref. [16] and Box 4.1 D of ref. [17], the invariance of charge, stated by equation (3.1), is strictly related to the continuity equation of charge (2.3).

4. The non–zero electric field hypotesis

Even if in [1] the questions of the exact way to interpret the charge invariance and of the existence of an $\boldsymbol{E} \neq 0$ externally to a current–carrying conductor may seem to be related, actually they are not (as stressed in [4]).

Historically, the existence of an $\boldsymbol{E} \neq 0$ outside a current–carrying conductor was largely discussed in the literature (see, for instance, [5] sect. **1**). Even if the usual belief is that such an $\boldsymbol{E}$ does not exist, however there is no experimental evidence. This question is not settled in the framework of the usual Maxwell theory (where the charge is assumed not to depend on the velocity); it can in fact happen that the positive charge density in a flowing current turns out to be different from that of negative charges.

According to [1] the problem of the existence of this field can be traced back to that of knowing in which frame $\lambda_+ = \lambda_-$. In [1,4] and [18] it is clearly stressed that two physical hypoteses are relevant to this point. The first one assumes that $\lambda_+ = \lambda_-$ in the wire frame, and this gives $\boldsymbol{E} = 0$ according to the common belief; the second one assumes that λ_+ (as evaluated in the ions rest frame) is equal to λ_- (as evaluated in the electrons rest frame). Ivezic prefers the latter because the charges are treated in a symmetric way. He also believes that some experimental results confirm it [5].

According to Ivezić view one expects a radial field $E = \delta\lambda/(2\pi r)$ where $\delta\lambda$ is the absolute value of the difference of charge density in the rest frame of the lattice and r is the radial distance from the wire. It turns out that

$$\delta\lambda = (\gamma - 1)\lambda_+ \simeq \frac{1}{2}\left(\frac{v}{c}\right)^2 \lambda_+ \tag{4.1}$$

where v is the drift velocity and, as usual, $\gamma = 1/\sqrt{1 - v^2/c^2}$. The field $\boldsymbol{E}$ is a second order quantity in v/c and therefore extremely small under ordinary conditions. This fact prevented from verifying its existence until now. One can hope that experiments with superconducting materials like those reported in [5] can settle the question. Such experiments have been however proposed not to this end but to measure possible second–order deviation from Maxwell's equations due to an hypothetical dependence of the charge on the velocity; in fact it was taken for granted by these experimentalists that ions and electrons charge densities where the the same in the rest frame of the ions. This same assumption was made in the paper by Baker [19] who has shown, starting from Lienard-Wiechert potentials, that there is no electric field produced by the charge drift in a current–carrying conductor in accordance with Gauss's law.

Ivezić in [1] and [4] raised the question of the possible existence of $\boldsymbol{E} \neq 0$ outside a current–carrying conductor in the framework of Maxwell's equations without postulating a charge dependence on the velocity as proposed in [5] but assuming $\lambda_+ \neq \lambda_-$ in ions frame.

In our opinion there are at least three reasons why the two densities should be equal in the ions frame.

The first one is related with thermal motion. Already in [12], Singal asked why Ivezić did not take into consideration the effects of thermal noise of electrons. In his words this means that "it is not clear how in Ivezić's approach the effects of the thermal velocities of electrons, many orders of magnitude larger than their drift velocities, could in some unambiguous way be incorporated or at least shown to cancel, since his derived electric fields (see eq. (4.1)) depend upon the square of the velocity of moving charges." It is clear that if we consider an isolated conductor its charge cannot vary. But let us consider the case of a wire that connects two charge reservoirs. Let us suppose that the distribution of the velocity of electrons at a certain temperature T (ions are considered at rest because of their large mass) is given by a function $f_T(v) = f_T(-v)$ normalized to unity (that is to say $\int_{\Re} f_T(v)dv = 1$). Let us call $O(v)$ the reference frames in motion with velocity v with respect to ions. If Ivezić ideas are correct, in that frame the charge density of electrons with velocity lying in the interval $(v, v+dv)$ must be equal to that of the ions in $O(0)$ with the opposite sign. This means that in the ions rest frame these electrons are characterized by a linear charge density $\lambda(v) = -\gamma(v)\lambda_+$. In this way the linear negative–charge density would be given by

$$\Lambda(T) = \int_{-\infty}^{+\infty} \lambda(v) f_T(v) dv \tag{4.2}$$

where Λ depends only on the temperature T. This means that there is an excess of negative charge per unit length of the amount $\delta\lambda(T) = \lambda_+ + \Lambda(T)$. If we consider ordinary temperature, the gas of electrons in the conductor is almost completely degenerate. In this way we can give an extimation of $f_T(v)$ independent of the temperature. To get the order of magnitude we can put

$$f(v) = \frac{1}{2v_F} \qquad v \leq |v_F| \tag{4.3}$$

where v_F is the Fermi velocity. In this way up to the second order in (v_F/c)

$$\Lambda \simeq -\left[1 + \frac{1}{6}\left(\frac{v_F}{c}\right)^2\right]\lambda_+. \tag{4.4}$$

The linear charge excess is then given by

$$\delta\lambda \simeq -\frac{\lambda_+}{6}\left(\frac{v_F}{c}\right)^2. \tag{4.5}$$

For a centimeter of copper wire with a cross section $S = 10^{-4}\ cm^2$, taking into account that $v_F = 1.56 \times 10^8\ cm/sec$ and $\lambda_+ = 8.5 \times 10^{22} eS\ C/cm$ (where e is the absolute value of the electron charge) one has a charge $Q = 6 \times 10^{-6} C$!

A second way to look at the problem of charge density is to consider the current flowing in the wire as due to an acceleration of electrons at rest in the wire caused by an applied electric field. The steady state is reached because of the inner resistance of the wire. Now all the electrons undergo the same accelerating field and therefore, assuming the same initial velocity (in this case equal to zero), their distance cannot change in the ions frame (cf. chapt. 20 in ref [16]). This means by the way that during the acceleration the proper distance among electrons increases due to the Lorentz contraction.

As a third consideration about Ivezić problem, we take into account Ohm's law. In the interior of a conductor at rest one has

$$\boldsymbol{j} = \sigma \boldsymbol{E} \tag{4.6}$$

where σ is the conductivity. By means of Gauss's law $div\boldsymbol{E} = 4\pi\rho$ and the continuity equation, one easily finds an equation for the density of the charge inside a conductor

$$\frac{\partial \rho}{\partial t} + 4\pi\sigma\rho = 0 \tag{4.7}$$

The solution of this equation is

$$\rho = \rho_0 exp\left(-\frac{t}{\tau}\right) \tag{4.8}$$

where $\tau = (4\pi\sigma)^{-1} \simeq 10^{-18}$ *sec* that is a time which is correct to take equal to zero in a macroscopic theory. This implies that conduction electrons inside a conductor have always the same density as ions. Moreover, for steady currents $0 = div\boldsymbol{j} = 4\pi\sigma\rho$, that is to say $\rho = 0$. Therefore, Ohm's law requires that inside a conducting wire the two charge densities are equal in the reference frame of ions. A possible charge excess, as that assumed by Ivezić, can only stay on the surface and this destroys the symmetry reason invoked by him.

Therefore, it appears reasonable that the birth of a current does not modify the electron charge density in the frame in which ions are at rest (and in which also the electrons were at rest before an electric field was applied).

In conclusion, even though Ivezić has the merit to have stressed that both situations (i.e. charge densities equal in the frame of the wire or in their own reference frame) are compatible with Maxwell's equation, the above considerations (which lie outside the domain of this theory) show however that the first of the two situation envisaged, *i.e.* the one commonly assumed, is the right one.

5. Conclusions

We have shown that the question raised by Ivezić about the non–invariance of the charge under Lorentz trasformations was originated by a misunderstanding of the usual charge definition. If such definition is properly understood, one has the usual charge invariance and there is no necessity of looking for new invariant charges as claimed by Ivezić.

As to the ambiguity pointed out by Ivezić about the reference frame in which the positive charge density should be equal to the negative one in a conductor carrying a stationary current, we have shown that the usual belief, that is to say that $\lambda_+ = \lambda_-$ in ions rest frame, is the correct one. In fact, if one assumes Ivezić's idea, the following three facts, clearly in contrast with experimental results, should happen: 1) there should be present very big effects due to the thermal motion of the electrons; 2) the same accelerating field on the same particles with the same initial velocities would change the mean distance among them; 3) Ohm's law would be no longer valid.

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